Department of Electrical and Computer Engineering

MMSE Estimator for linear non-Gaussian Dynamic Systems

Leila Pishdad

Fabrice Labeau

Introduction

Suppose a discrete-time linear dynamic state space system with the following state and measurement equations:

$$\mathbf{x}_k = F_k \mathbf{x}_{k-1} + \mathbf{v}_k, \quad (1)$$

$$\mathbf{z}_k = H_k \mathbf{x}_k + \mathbf{w}_k, \quad (2)$$

Where:

- $\{\mathbf{x}_k, k \in \mathbb{N}\}$ is the unobservable state sequence
- $\{\mathbf{z}_k, k \in \mathbb{N}\}$ is the observable measurement sequence
- $\{\mathbf{v}_k, k \in \mathbb{N}\}$ and $\{\mathbf{w}_k, k \in \mathbb{N}\}$, are i.i.d. random vectors with known pdfs $p(\mathbf{v}_k)$ and $p(\mathbf{w}_k)$
- F_k and H_k are known matrices

The objective of Bayesian tracking is to find a probabilistic estimation of the unobservable current state, given the available measurements. This is done by recursively estimating the posterior, $p(\mathbf{x}_k|\mathbf{z}_{1:k})$. If the noise vectors $\mathbf{v}_k$ and $\mathbf{w}_k$, are Gaussian, the posterior is Gaussian, and its parameters are optimally estimated by a Kalman filter.

Non-Gaussian Noise

In this work we have used Gaussian Mixtures to model the non-Gaussian noise pdfs, since they can approximate the distribution as closely as desired and this estimation is asymptotically unbiased. Additionally, they are mathematically tractable. Hence, the process noise and measurement noises can be approximated as

$$p(\mathbf{v}_k) \approx \sum_{i=1}^{C_{\mathrm{v}_k}} w_k^i \, \mathcal{N}(\mathbf{v}_k; \mathbf{u}_k^i, Q_k^i), \quad (3)$$

$$p(\mathbf{w}_k) \approx \sum_{j=1}^{C_{\mathrm{w}_k}} \gamma_k^j \, \mathcal{N}(\mathbf{w}_k; \mathbf{b}_k^j, R_k^j), \quad (4)$$

Using the GMM approximations the dynamic system defined in Eq. (1)-(2), can be described as a multiple model system, with $\left\{\mathrm{M}_k^{ij}, 1 \le i \le C_{\mathrm{v}_k}, 1 \le j \le C_{\mathrm{w}_k}\right\}$ representing the different clusters of the process and measurement noises. Hence, the posterior can be partitioned as

$$\mathrm{p}(\mathrm{x}_k|\mathrm{z}_{1:k}) = \sum_{i,j} \mu^{ij}\, \mathrm{p}(\mathrm{x}_k|\mathrm{z}_{1:k}, \mathrm{M}_k^{ij}), \quad (5)$$

where, $\mathrm{p}\left(\mathrm{x}_k|\mathrm{z}_{1:k}, \mathrm{M}_k^{ij}\right)$ can be approximated by a Gaussian, since it has a mode-conditioned Gaussian process and measurement noise, and $\mu^{ij} = \mathrm{p}(\mathrm{M}_k^{ij}|\mathrm{z}_{1:k})$.

Gaussian Sum Filters

In Gaussian Sum Filters, $\mathrm{p}(\mathrm{x}_k|\mathrm{z}_{1:k}, \mathrm{M}_k^{ij})$ in Eq. (5) is approximated using a Kalman filter. Hence, the filter is composed of $C_{\mathrm{v}_k} C_{\mathrm{w}_k}$ parallel Kalman filters, corresponding to the modes of the posterior pdf in Eq. (5). If we denote the estimated state and covariance matrix of filter ij, by $\hat{\mathbf{x}}_{k|k}^{ij}$ and $P_{k|k}^{ij}$, respectively, the total estimated state and estimation error covariance matrix can be evaluated as:

$$\hat{\mathbf{x}}_{k|k} \approx \sum_{i,j} \mu^{ij}\, \hat{\mathbf{x}}_{k|k}^{ij}, \quad (6)$$

$$P_{k|k} \approx \sum_{i,j} \mu^{ij} \left(P_{k|k}^{ij} + \hat{\mathbf{x}}_{k|k}^{ij} \hat{\mathbf{x}}_{k|k}^{ij\,T}\right) - \hat{\mathbf{x}}_{k|k} \hat{\mathbf{x}}_{k|k}^{T}, \quad (7)$$

Kalman filter provides the MMSE state estimation, as it minimizes the trace of the state estimation error covariance matrix. However, we can see that for GSF, the total estimation error covariance (Eq. (7)) also has the spread of means of the clusters which is not minimized with the Kalman gains W_k^{ij}.

MMSE Estimator

The MMSE estimator is a modification of GSF in the sense that the individual filter gains, $\boldsymbol{W}_{\boldsymbol{k}}^{\boldsymbol{ij}}$ are evaluated such that the trace of the total estimation error covariance matrix in Eq. (7) is minimized. This is done by writing

$$P_{k|k}^{ij} = P_{k|k-1}^{ij} - W_k^{ij} H_k P_{k|k-1}^{ij} - P_{k|k-1}^{ij} H_k^T W_k^{ij\,T} + W_k^{ij} S_k^{ij} W_k^{ij\,T}, \quad (8)$$

$$\hat{\mathbf{x}}_{k|k}^{ij} = F_k \hat{\mathbf{x}}_{k|k-1}^{ij} + W_k^{ij}\left(\mathbf{z}_k - \hat{\mathbf{z}}_k^{ij}\right), \quad (9)$$

where, $\hat{\mathbf{x}}_{k|k-1}^{ij}$, $P_{k|k-1}^{ij}$, $\hat{\mathbf{z}}_k^{ij}$, and S_k^{ij} are the predicted state, state prediction covariance matrix, predicted measurement, and measurement prediction covariance matrix respectively, and are evaluated using Kalman's prediction step of filter $\boldsymbol{ij}$. Using Eq. (8)-(9) in Eq. (7), and setting $\frac{\partial \mathrm{tr}(P_{k|k})}{\partial W_k^{ij}} = 0$, we have $W_k^{ij\,*} = \left(P_{k|k-1}^{ij} H_k^T + \left(\sum_{l,m} \mu^{lm}\, \mathbf{u}_k^l - \mathbf{u}_k^i\right)\left(\mathbf{z}_k - \hat{\mathbf{z}}_k^{ij}\right)^T + \mathbb{S}_k\left(\mathbf{z}_k - \hat{\mathbf{z}}_k^{ij}\right)\right)\left(S_k^{ij}\left(\mathbf{z}_k - \hat{\mathbf{z}}_k^{ij}\right)\left(\mathbf{z}_k - \hat{\mathbf{z}}_k^{ij}\right)^T\right)$, where $\mathbb{S}_k = \sum_{l,m} \mu^{lm}\, W_k^{lm\,*}$.

Simulation Results

The proposed filtering method is compared with GSF, by running simulations on synthetically generated data (Figure(1)) and experimental data (Figure(2)). The performance metrics used for comparison are Root-Mean-Square Error (RMSE) and Circular Error Probable (CEP). Since the number of posterior modes increase exponentially over time, after each update we remove the clusters with smaller weights.

Table 1: RMSE and CEP for GSF and MMSE estimator

Filter	Data	RMSE	CEP
MMSE estimator	Synthetic	0.9144	0.6832
GSF	synthetic	1.4380	1.0406
MMSE estimator	Experimental	42.5749	20.3706
GSF	Experimental	204.1124	178.3923

The MMSE estimator shows better performance when compared with GSF, as it minimizes the total state estimation covariance, including the spread of means. Consequently, in GSF the variance of cluster means is larger.

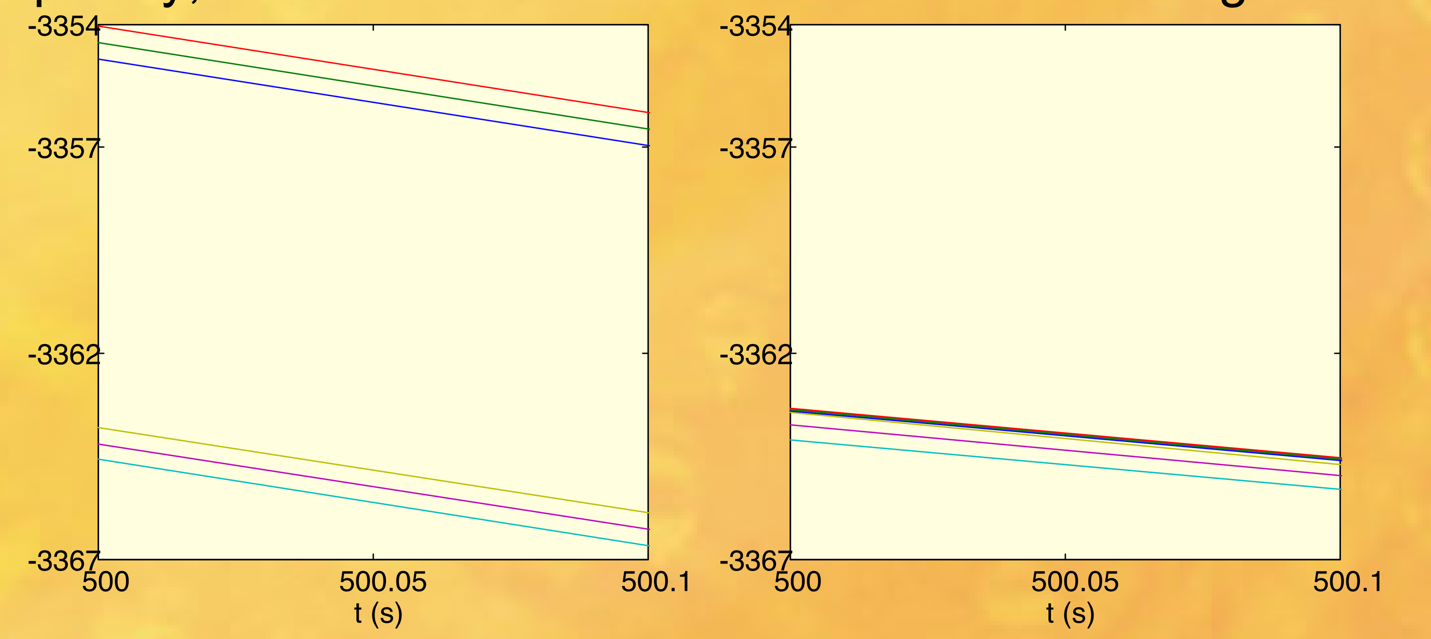

Figure 1: Cluster centers for GSF (left) and MMSE filter (right) for synthetic data

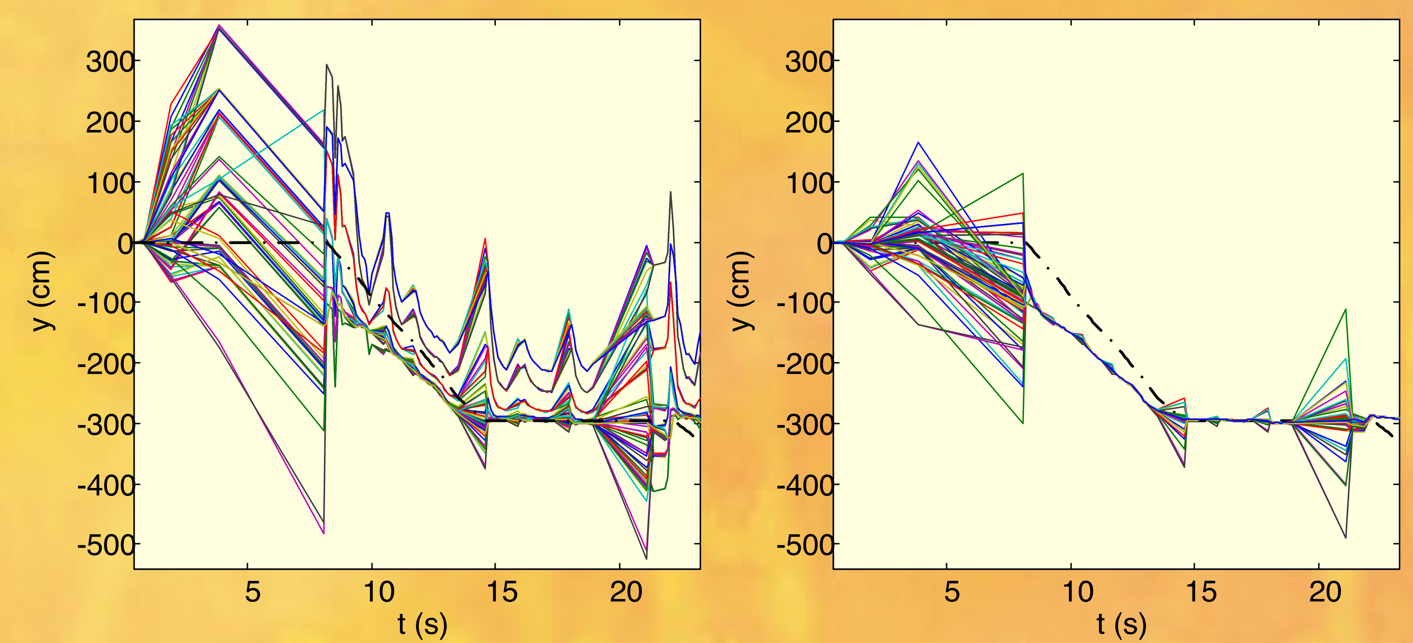

Figure 2:Cluster centers for GSF (left) and MMSE filter (right) for experimental data